APPENDIX A
DIMENSIONLESS EQUATIONS IN CURVILINEAR BOUNDARY-FITTED AND SIGMA GRID

Non-dimensionalization of governing equations make it easy to compare the relative importance of various terms in the equations. The governing equations are non-dimensionalized using the following reference scales: X_r and Z_r are the reference lengths in the vertical and horizontal directions; U_r is the reference velocity; ρ, ρ_r and $\Delta\rho = \rho - \rho_0$ are the reference density, mean density and density gradient in a stratified flow; A_{Hr} and A_{Vr} are the reference eddy viscosities in the horizontal and vertical directions; D_{Hr} and D_{Vr} are the reference eddy diffusivities in the horizontal and vertical directions. The dimensionless variables can be written as (Sheng, 1983)

$$\left(x^*, y^*, z^*\right) = \left(x, y, zX_r / Z_r\right) / X_r$$

$$\left(u^*, v^*, w^*\right) = \left(u, v, wX_r / Z_r\right) / U_r$$

$$\omega^* = \omega X_r / U_r$$

$$t^* = tf \tag{A.1}$$

$$\left(\tau_x^*, \tau_y^*\right) = \left(\tau_x^w, \tau_y^w\right) / \left(\rho_0 f Z_r U_r\right) = \left(\tau_x^w, \tau_y^w\right) / \tau_r$$

$$\zeta^* = g\zeta / \left(fU_r X_r\right) = \zeta / S_r$$

$$A_H^* = A_H / A_{Hr}$$

$$A_V^* = A_V / A_{Vr}$$

$$D_H^* = D_H / D_{Hr}$$

$$D_V^* = D_V / D_{Vr}$$

These dimensionless variables can be combined to yield the following dimensionless parameters

Rossby number : $R_0 = \frac{U_r}{\sqrt{gZ_r}}$

Froude Number : $F_r = \frac{U_r}{\sqrt{gZ_r}}$

Densimetric Froude Number :

$$F_{rd} = \frac{F_r}{\sqrt{\varepsilon}} \qquad \text{(A.2)}$$

Vertical and Horizontal Ekman Number : $E_V = \frac{A_{Vr}}{fZ_r^2}, E_H = \frac{A_{Hr}}{fX_r^2}$

Vertical and Horizontal Schmidt Number : $S_{CV} = \frac{A_{Vr}}{D_{Vr}}, S_{CH} = \frac{A_{Hr}}{D_{Hr}}$

$$\varepsilon = \frac{\rho_r - \rho_0}{\rho_0}$$

$$\beta = \frac{gZ_r}{f^2 X_r^2} = \left(\frac{R_0}{F_r}\right)^2$$

In three-dimensional modeling, complex bottom topographies can be better represented with the application of σ-stretching (Sheng, 1983). This transformation allows the same vertical resolution in the shallow coastal areas and the deeper navigation channels. The vertical coordinate, z, is transformed into a new coordinate, σ, by (Phillips, 1957).

$$\sigma = \frac{z - \zeta(x, y, t)}{h(x, y) + \zeta(x, y, t)} \tag{A.3}$$

where h is the water depth and ς is the surface elevation.

In this new vertical coordinate, the vertical velocity is calculated by the following equation.

$$w = H\omega + (1+\sigma)\frac{D\zeta}{Dt} + \sigma\left(u\frac{dh}{dx} + v\frac{dh}{dy}\right) \tag{A.4}$$

where $w = \frac{dz}{dt}$ in the z-plane, $\omega = \frac{d\sigma}{dt}$ in the σ-plane

Using non-orthogonal boundary-fitted horizontal grid, it is possible to better represent the circulation and transport processes in estuarine systems with complex shoreline geometries. Using the elliptic grid generation technique developed by Thompson (1982) and Thompson et al. (1985), a non-orthogonal boundary-fitted grid can be generated in the horizontal dimension. To solve for flow in a boundary-fitted grid, it is necessary to transform the governing equations from original coordinates (x, y) to the transformed coordinates (ξ, η). The spatial coordinate system in the computational plane (ξ, η) is dimensionless while the coordinates in the physical plane (x, y) have dimensions of length. During the transformations, the velocities are transformed into contra-variant velocities.

In the boundary-fitted, curvilinear, σ-stretched, non-dimensional coordinate system, the continuity and momentum equations are

$$\frac{\partial \zeta}{\partial t}+\frac{\beta}{\sqrt{g_0}}\left[\frac{\partial}{\partial \xi}\left(\sqrt{g_0}Hu\right)+\frac{\partial}{\partial \eta}\left(\sqrt{g_0}Hv\right)\right]+\beta\frac{\partial H\omega}{\partial \sigma}=0 \qquad \text{(A.6)}$$

$$\begin{aligned}
\frac{1}{H}\frac{\partial Hu}{\partial t} = & -\left(g^{11}\frac{\partial \zeta}{\partial \xi}+g^{12}\frac{\partial \zeta}{\partial \eta}\right)+\left(\frac{g^{12}}{\sqrt{g_0}}u+\frac{g^{22}}{\sqrt{g_0}}v\right) \\
& +\frac{R_0}{g_0 H}\left\{x_\eta\left[\frac{\partial}{\partial \xi}\left(y_\xi\sqrt{g_0}Huu+y_\eta\sqrt{g_0}Huv\right)+\frac{\partial}{\partial \eta}\left(y_\xi\sqrt{g_0}Huv+y_\eta\sqrt{g_0}Hvv\right)\right]\right. \\
& \qquad -y_\eta\left[\frac{\partial}{\partial \xi}\left(x_\xi\sqrt{g_0}Huu+x_\eta\sqrt{g_0}Huv\right)+\frac{\partial}{\partial \eta}\left(x_\xi\sqrt{g_0}Huv+x_\eta\sqrt{g_0}Hvv\right)\right] \\
& \qquad \left.-g_0\frac{\partial Huw}{\partial \sigma}\right\} \\
& -\frac{R_0}{F_r^2}\left[H\int_\sigma^0\left(g^{11}\frac{\partial \rho}{\partial \xi}+g^{12}\frac{\partial \rho}{\partial \eta}\right)d\sigma+\left(g^{11}\frac{\partial H}{\partial \xi}+g^{12}\frac{\partial H}{\partial \eta}\right)\left(\int_\sigma^0\rho d\sigma+\sigma\rho\right)\right] \\
& +\frac{E_V}{H^2}\frac{\partial}{\partial \sigma}\left(A_V\frac{\partial u}{\partial \sigma}\right) \quad +E_H A_H(Horizontal \quad Diffusion \quad of \quad u)
\end{aligned} \qquad \text{(A.7)}$$

$$\begin{aligned}
\frac{1}{H}\frac{\partial Hv}{\partial t} = & -\left(g^{21}\frac{\partial \zeta}{\partial \xi}+g^{22}\frac{\partial \zeta}{\partial \eta}\right)+\left(\frac{g^{11}}{\sqrt{g_0}}u+\frac{g^{21}}{\sqrt{g_0}}v\right) \\
& +\frac{R_0}{g_0 H}\left\{x_\xi\left[\frac{\partial}{\partial \xi}\left(y_\xi\sqrt{g_0}Huv+y_\eta\sqrt{g_0}Hvv\right)+\frac{\partial}{\partial \eta}\left(y_\xi\sqrt{g_0}Huv+y_\eta\sqrt{g_0}Hvv\right)\right]\right. \\
& \qquad -y_\xi\left[\frac{\partial}{\partial \xi}\left(x_\xi\sqrt{g_0}Huu+x_\eta\sqrt{g_0}Huv\right)+\frac{\partial}{\partial \eta}\left(x_\xi\sqrt{g_0}Huv+x_\eta\sqrt{g_0}Hvv\right)\right] \\
& \qquad \left.-g_0\frac{\partial Hvw}{\partial \sigma}\right\} \\
& -\frac{R_0}{F_r^2}\left[H\int_\sigma^0\left(g^{21}\frac{\partial \rho}{\partial \xi}+g^{22}\frac{\partial \rho}{\partial \eta}\right)d\sigma+\left(g^{21}\frac{\partial H}{\partial \xi}+g^{22}\frac{\partial H}{\partial \eta}\right)\left(\int_\sigma^0\rho d\sigma+\sigma\rho\right)\right] \\
& +\frac{E_V}{H^2}\frac{\partial}{\partial \sigma}\left(A_V\frac{\partial v}{\partial \sigma}\right) \quad +E_H A_H(Horizontal \quad Diffusion \quad of \quad v)
\end{aligned} \qquad \text{(A.8)}$$

$$\sqrt{g_0} = J = x_\xi y_\eta - x_\eta y_\xi \tag{A.9}$$

is the determinant of the matric tensor, g_{ij} , which is defined as

$$g^{ij} = \begin{bmatrix} x_\xi^2 + y_\xi^2 & x_\xi x_\eta + y_\xi y_\eta \\ x_\eta x_\xi + y_\eta y_\xi & x_\eta^2 + y_\eta^2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \tag{A.10}$$

whose inverse is

$$g^{ij} = \begin{bmatrix} x_\eta^2 + y_\eta^2 & -(x_\xi x_\eta + y_\xi y_\eta) \\ -(x_\eta x_\xi + y_\eta y_\xi) & x_\xi^2 + y_\xi^2 \end{bmatrix} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \tag{A.11}$$

As shown in Sheng (1986), the contravarient components (u^i) and physical components ($u(I)$) of the velocity vector in the non-Cartesian system are locally parallel or orthogonal to the grid lines, while the covarient components (u_i) are generally not parallel or orthogonal to the local grid lines. The relationship between the physical velocity and contravarient velocity is given by (Sheng, 1986)

$$u^i = \frac{u(i)}{\sqrt{g_{ii}}} \tag{A.12}$$

with summation on i.

The salinity and temperature transport equations can be written as

$$\frac{\partial HS}{\partial t}=\frac{E_V}{H\cdot S_{CV}}\frac{\partial}{\partial\sigma}\left(D_V\frac{\partial S}{\partial\sigma}\right)\quad -R_0\frac{\partial HwS}{\partial\sigma}$$
$$-\frac{R_0}{\sqrt{g_0}}\left[\frac{\partial}{\partial\xi}\left(\sqrt{g_0}HuS\right)+\frac{\partial}{\partial\eta}\left(\sqrt{g_0}HvS\right)\right]$$
$$+\frac{E_V}{S_{CH}\sqrt{g_0}}\left[\frac{\partial}{\partial\xi}\left(\sqrt{g_0}Hg^{11}\frac{\partial S}{\partial\xi}+\sqrt{g_0}Hg^{12}\frac{\partial S}{\partial\eta}\right)\right] \tag{A.13}$$
$$+\frac{E_V}{S_{CH}\sqrt{g_0}}\left[\frac{\partial}{\partial\eta}\left(\sqrt{g_0}Hg^{21}\frac{\partial S}{\partial\xi}+\sqrt{g_0}Hg^{22}\frac{\partial S}{\partial\eta}\right)\right]$$

and

$$\frac{\partial HT}{\partial t}=\frac{E_V}{H\cdot S_{CV}}\frac{\partial}{\partial\sigma}\left(D_V\frac{\partial T}{\partial\sigma}\right)\quad -R_0\frac{\partial HwT}{\partial\sigma}$$
$$-\frac{R_0}{\sqrt{g_0}}\left[\frac{\partial}{\partial\xi}\left(\sqrt{g_0}HuT\right)+\frac{\partial}{\partial\eta}\left(\sqrt{g_0}HvT\right)\right]$$
$$+\frac{E_V}{S_{CH}\sqrt{g_0}}\left[\frac{\partial}{\partial\xi}\left(\sqrt{g_0}Hg^{11}\frac{\partial T}{\partial\xi}+\sqrt{g_0}Hg^{12}\frac{\partial T}{\partial\eta}\right)\right] \tag{A.14}$$
$$+\frac{E_V}{S_{CH}\sqrt{g_0}}\left[\frac{\partial}{\partial\eta}\left(\sqrt{g_0}Hg^{21}\frac{\partial T}{\partial\xi}+\sqrt{g_0}Hg^{22}\frac{\partial T}{\partial\eta}\right)\right]$$

The sediment transport equation can be written as

$$\frac{\partial Hc_i}{\partial t}=\frac{E_V}{H\cdot S_{CV}}\frac{\partial}{\partial\sigma}\left(D_V\frac{\partial c_i}{\partial\sigma}\right)\quad -R_0\frac{\partial(H\omega-w_{si})c_i}{\partial\sigma}$$
$$-\frac{R_0}{\sqrt{g_0}}\left[\frac{\partial}{\partial\xi}\left(\sqrt{g_0}Huc_i\right)+\frac{\partial}{\partial\eta}\left(\sqrt{g_0}Hvc_i\right)\right]$$
$$+\frac{E_V}{S_{CH}\sqrt{g_0}}\left[\frac{\partial}{\partial\xi}\left(\sqrt{g_0}Hg^{11}\frac{\partial c_i}{\partial\xi}+\sqrt{g_0}Hg^{12}\frac{\partial c_i}{\partial\eta}\right)\right] \tag{A.15}$$
$$+\frac{E_V}{S_{CH}\sqrt{g_0}}\left[\frac{\partial}{\partial\eta}\left(\sqrt{g_0}Hg^{21}\frac{\partial c_i}{\partial\xi}+\sqrt{g_0}Hg^{22}\frac{\partial c_i}{\partial\eta}\right)\right]$$

where c_i represents cohesive (i=1) and non-cohesive (i=2) sediment concentrations and w_{si} is settling velocity for sediment group i.