APPENDIX A
DIMENSIONLESS EQUATIONS IN CURVILINEAR BOUNDARY-FITTED AND SIGMA GRID

Non-dimensionalization of governing equations make it easy to compare the relative importance of various terms in the equations. The governing equations are non-dimensionalized using the following reference scales: \( X_r \) and \( Z_r \) are the reference lengths in the vertical and horizontal directions; \( U_r \) is the reference velocity; \( \rho, \rho_r \) and \( \Delta \rho = \rho - \rho_0 \) are the reference density, mean density and density gradient in a stratified flow; \( A_{hr} \) and \( A_{vr} \) are the reference eddy viscosities in the horizontal and vertical directions; \( D_{hr} \) and \( D_{vr} \) are the reference eddy diffusivities in the horizontal and vertical directions. The dimensionless variables can be written as (Sheng, 1983)

\[
\begin{align*}
\left( x^*, y^*, z^* \right) &= \left( x, y, zX_r / Z_r \right) / X_r \\
\left( u^*, v^*, w^* \right) &= \left( u, v, wX_r / Z_r \right) / U_r \\
\omega^* &= \omega X_r / U_r \\
t^* &= tf \\
\left( \tau_x^*, \tau_y^* \right) &= \left( \tau_x^w, \tau_y^w \right) \left( \rho_0 fZ_r U_r \right) = \left( \tau_x^w, \tau_y^w \right) / \tau_r \\
\zeta^* &= g \zeta / \left( fU_r X_r \right) = \zeta / S_r
\end{align*}
\]
These dimensionless variables can be combined to yield the following dimensionless parameters

**Rossby number:**

\[ R_0 = \frac{U_r}{\sqrt{gZ_r}} \]

**Froude Number:**

\[ F_r = \frac{U_r}{\sqrt{gZ_r}} \]

**Densimetric Froude Number:**

\[ F_{rd} = \frac{F_r}{\sqrt{\varepsilon}} \quad (A.2) \]

**Vertical and Horizontal Ekman Number:**

\[ E_v = \frac{A_{v_r}}{fZ_r^2}, E_H = \frac{A_{Hr}}{fX_r^2} \]

**Vertical and Horizontal Schmidt Number:**

\[ S_{CV} = \frac{A_{v_r}}{D_{v_r}}, S_{CH} = \frac{A_{Hr}}{D_{Hr}} \]

\[ \varepsilon = \frac{\rho_r - \rho_0}{\rho_0} \]

\[ \beta = \frac{gZ_r}{f^2X_r^2} = \left( \frac{R_0}{F_r} \right)^2 \]
In three-dimensional modeling, complex bottom topographies can be better represented with the application of $\sigma$-stretching (Sheng, 1983). This transformation allows the same vertical resolution in the shallow coastal areas and the deeper navigation channels. The vertical coordinate, $z$, is transformed into a new coordinate, $\sigma$, by (Phillips, 1957).

$$\sigma = \frac{z - \zeta(x,y,t)}{h(x,y) + \zeta(x,y,t)}$$  \hspace{1cm} (A.3)

where $h$ is the water depth and $\zeta$ is the surface elevation.

In this new vertical coordinate, the vertical velocity is calculated by the following equation.

$$w = H \omega + (1 + \sigma) \frac{D\zeta}{Dt} + \sigma \left( u \frac{dh}{dx} + v \frac{dh}{dy} \right)$$  \hspace{1cm} (A.4)

where $w = \frac{dz}{dt}$ in the $z$-plane, $\omega = \frac{d\sigma}{dt}$ in the $\sigma$-plane.

Using non-orthogonal boundary-fitted horizontal grid, it is possible to better represent the circulation and transport processes in estuarine systems with complex shoreline geometries. Using the elliptic grid generation technique developed by Thompson (1982) and Thompson et al. (1985), a non-orthogonal boundary-fitted grid can be generated in the horizontal dimension. To solve for flow in a boundary-fitted grid, it is necessary to transform the governing equations from original coordinates $(x, y)$ to the transformed coordinates $(\xi, \eta)$. The spatial coordinate system in the computational plane $(\xi, \eta)$ is dimensionless while the coordinates in the physical plane $(x, y)$ have dimensions of length. During the transformations, the velocities are transformed into contra-variant velocities.

In the boundary-fitted, curvilinear, $\sigma$-stretched, non-dimensional coordinate system, the
continuity and momentum equations are

\[
\frac{\partial \xi}{\partial t} + \frac{\beta}{\sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} \text{Hu} \right) + \frac{\partial}{\partial \eta} \left( \sqrt{g_0} \text{Hv} \right) \right] + \beta \frac{\partial \text{H} \omega}{\partial \sigma} = 0
\]  
(A.6)

\[
\frac{1}{H} \frac{\partial \text{Hu}}{\partial t} = -\left( g^{11} \frac{\partial \xi}{\partial \xi} + g^{12} \frac{\partial \xi}{\partial \eta} \right) + \left( g^{12} \frac{\partial u}{\sqrt{g_0}} + g^{22} \frac{\partial v}{\sqrt{g_0}} \right) \\
\quad + \frac{R_0}{g_0 H} \left[ x_\eta \left[ \frac{\partial}{\partial \xi} \left( y_\xi \sqrt{g_0} \text{Hu} + y_\eta \sqrt{g_0} \text{Hv} \right) + \frac{\partial}{\partial \eta} \left( y_\xi \sqrt{g_0} \text{Huv} + y_\eta \sqrt{g_0} \text{Huv} \right) \right] \\
\quad - y_\eta \left[ \frac{\partial}{\partial \xi} \left( x_\xi \sqrt{g_0} \text{ Huu} + x_\eta \sqrt{g_0} \text{Huv} \right) + \frac{\partial}{\partial \eta} \left( x_\xi \sqrt{g_0} \text{Hu} + x_\eta \sqrt{g_0} \text{Hv} \right) \right] \\
\quad - g_0 \frac{\partial \text{Huv}}{\partial \sigma} \right] \\
\quad + \frac{E_v}{H^2} \frac{\partial}{\partial \sigma} \left( A_v \frac{\partial u}{\partial \sigma} \right) + E_{Hv} (\text{Horizontal Diffusion of } u)
\]  
(A.7)

\[
\frac{1}{H} \frac{\partial \text{Hv}}{\partial t} = -\left( g^{21} \frac{\partial \xi}{\partial \xi} + g^{22} \frac{\partial \xi}{\partial \eta} \right) + \left( g^{12} \frac{\partial u}{\sqrt{g_0}} + g^{21} \frac{\partial v}{\sqrt{g_0}} \right) \\
\quad + \frac{R_0}{g_0 H} \left[ x_\xi \left[ \frac{\partial}{\partial \xi} \left( y_\xi \sqrt{g_0} \text{Huv} + y_\eta \sqrt{g_0} \text{Huv} \right) + \frac{\partial}{\partial \eta} \left( y_\xi \sqrt{g_0} \text{Huv} + y_\eta \sqrt{g_0} \text{Huv} \right) \right] \\
\quad - y_\eta \left[ \frac{\partial}{\partial \xi} \left( x_\xi \sqrt{g_0} \text{Huv} + x_\eta \sqrt{g_0} \text{Huv} \right) + \frac{\partial}{\partial \eta} \left( x_\xi \sqrt{g_0} \text{Huv} + x_\eta \sqrt{g_0} \text{Huv} \right) \right] \\
\quad - g_0 \frac{\partial \text{Huv}}{\partial \sigma} \right] \\
\quad + \frac{E_v}{H^2} \frac{\partial}{\partial \sigma} \left( A_v \frac{\partial v}{\partial \sigma} \right) + E_{Hv} (\text{Horizontal Diffusion of } v)
\]  
(A.8)
\sqrt{g_0} = J = x_\xi y_\eta - x_\eta y_\xi \tag{A.9}

is the determinant of the metric tensor, \( g_{ij} \), which is defined as

\[
\begin{bmatrix}
  x_\xi^2 + y_\xi^2 & x_\xi x_\eta + y_\xi y_\eta \\
  x_\eta x_\xi + y_\eta y_\xi & x_\eta^2 + y_\eta^2
\end{bmatrix}
= \begin{bmatrix}
  g_{11} & g_{12} \\
  g_{21} & g_{22}
\end{bmatrix} \tag{A.10}
\]

whose inverse is

\[
\begin{bmatrix}
  x_\eta^2 + y_\eta^2 & -(x_\xi x_\eta + y_\xi y_\eta) \\
  -(x_\eta x_\xi + y_\eta y_\xi) & x_\xi^2 + y_\xi^2
\end{bmatrix}
= \begin{bmatrix}
  g^{11} & g^{12} \\
  g^{21} & g^{22}
\end{bmatrix} \tag{A.11}
\]

As shown in Sheng (1986), the contravariant components \((u^i)\) and physical components \((u(I))\) of the velocity vector in the non-Cartesian system are locally parallel or orthogonal to the grid lines, while the covariant components \((u_i)\) are generally not parallel or orthogonal to the local grid lines. The relationship between the physical velocity and contravariant velocity is given by (Sheng, 1986)

\[
u^i = \frac{u(i)}{\sqrt{g_{ii}}} \tag{A.12}
\]

with summation on \(i\).

The salinity and temperature transport equations can be written as
\[
\frac{\partial HS}{\partial t} = \frac{E_v}{H \cdot S_{CV}} \frac{\partial}{\partial \sigma} \left( D_v \frac{\partial S}{\partial \sigma} \right) - R_0 \frac{\partial HwS}{\partial \sigma} \\
- \frac{R_0}{\sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} HuS \right) + \frac{\partial}{\partial \eta} \left( \sqrt{g_0} HvS \right) \right] \\
+ \frac{E_v}{S_{CH} \sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} Hg_{11} \frac{\partial S}{\partial \xi} + \sqrt{g_0} Hg_{12} \frac{\partial S}{\partial \eta} \right) \right] \\
+ \frac{E_v}{S_{CH} \sqrt{g_0}} \left[ \frac{\partial}{\partial \eta} \left( \sqrt{g_0} Hg_{21} \frac{\partial S}{\partial \xi} + \sqrt{g_0} Hg_{22} \frac{\partial S}{\partial \eta} \right) \right]
\] (A.13)

\[
\frac{\partial HT}{\partial t} = \frac{E_v}{H \cdot S_{CV}} \frac{\partial}{\partial \sigma} \left( D_v \frac{\partial T}{\partial \sigma} \right) - R_0 \frac{\partial HwT}{\partial \sigma} \\
- \frac{R_0}{\sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} HuT \right) + \frac{\partial}{\partial \eta} \left( \sqrt{g_0} HvT \right) \right] \\
+ \frac{E_v}{S_{CH} \sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} Hg_{11} \frac{\partial T}{\partial \xi} + \sqrt{g_0} Hg_{12} \frac{\partial T}{\partial \eta} \right) \right] \\
+ \frac{E_v}{S_{CH} \sqrt{g_0}} \left[ \frac{\partial}{\partial \eta} \left( \sqrt{g_0} Hg_{21} \frac{\partial T}{\partial \xi} + \sqrt{g_0} Hg_{22} \frac{\partial T}{\partial \eta} \right) \right]
\] (A.14)

The sediment transport equation can be written as

\[
\frac{\partial Hc_i}{\partial t} = \frac{E_v}{H \cdot S_{CV}} \frac{\partial}{\partial \sigma} \left( D_v \frac{\partial c_i}{\partial \sigma} \right) - R_0 \frac{\partial (H \omega - w_{si}) c_i}{\partial \sigma} \\
- \frac{R_0}{\sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} Huc_i \right) + \frac{\partial}{\partial \eta} \left( \sqrt{g_0} Hvc_i \right) \right] \\
+ \frac{E_v}{S_{CH} \sqrt{g_0}} \left[ \frac{\partial}{\partial \xi} \left( \sqrt{g_0} Hg_{11} \frac{\partial c_i}{\partial \xi} + \sqrt{g_0} Hg_{12} \frac{\partial c_i}{\partial \eta} \right) \right] \\
+ \frac{E_v}{S_{CH} \sqrt{g_0}} \left[ \frac{\partial}{\partial \eta} \left( \sqrt{g_0} Hg_{21} \frac{\partial c_i}{\partial \xi} + \sqrt{g_0} Hg_{22} \frac{\partial c_i}{\partial \eta} \right) \right]
\] (A.15)
where $c_i$ represents cohesive ($i=1$) and non-cohesive ($i=2$) sediment concentrations and $w_{si}$ is settling velocity for sediment group $i$. 